Numerical Integration with Rational Landen Transformations

Keywords: Landen transformation, numerical integration, rational functions.

In the summer of 2008, I worked with Prof. Victor Moll of Tulane University and two fellow undergraduates on the realization as a method for numerical integration of a technique that employs Landen transformations on rational integrands. I found the idea of using these transformations as a numerical tool novel and interesting. During that summer, I learned a lot about these and their possible use for integration, but there is still much I would like to investigate. I think that the possibilities that arise in furthering the success of this approach to integration require further research. And as there are several more approachable directions for research that are accessible to beginning math students, I could easily include such students in my research activities as suggested in my personal statement.

I will begin by introducing Landen transformations. A Landen transformation is a transformation on an integrand which preserves the value of an integral. Gauss studied one such transformation that has proven useful in calculating $\pi$ to high precision (see [1]). He observed that the elliptic integral $G(a, b)$ below is invariant under the transformation $\mathcal{E}$ below

$$G(a, b) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \quad \mathcal{E} : (a, b) \mapsto \left( \frac{a + b}{2}, \sqrt{ab} \right).$$

This is known as the elliptic Landen transformation. And, because the value of $G(a, b)$ is invariant under $\mathcal{E}$, it is invariant under the iterated transformation $\mathcal{E}^n$. For any $a, b$ the limit as $n$ goes to infinity of either component of $\mathcal{E}^n(a, b)$ is the same and is known as the arithmetic-geometric mean of $a$ and $b$, or more concisely AGM($a, b$). Thus, taking the limit as $\mathcal{E}$ is iterated, $\pi/(2 \text{AGM}(a, b)) = G(a, b)$. This was perhaps the first meaningful encounter with Landen transformations in mathematical history.

A rational analogue of this transformation was developed by Manna and Moll in [2]. For $p \in 2\mathbb{N}$, letting $\mathcal{R}_p$ denote rational functions of degree at most $p - 2$ over $p$ with convergent integral over the real line and $R \in \mathcal{R}_p$, a family of Landen transformations $\mathcal{L}_{m,p} : \mathcal{R}_p \rightarrow \mathcal{R}_p$ that leave $I = \int_{-\infty}^\infty R(x)\,dx$ invariant is introduced. The family of transformations is relevant to integration because of its important property that iterating $\mathcal{L}_{m,p}$ on $R$ yields rational functions that approach (coefficient-wise)

$$\int_{-\infty}^\infty \frac{dx}{1 + x^2} = \pi.$$

Thus if $\phi$ maps rational functions to their value at the origin (or, alternatively, to the ratio of their constant terms), then $\phi \circ \mathcal{L}^n_{m,p}(R) \rightarrow I/\pi$ as $n \rightarrow \infty$. Moreover, for each $m, p$ there is a constant $C$ such that $|\pi \phi \circ \mathcal{L}^n_{m,p}(R) - I| < C \pi \phi \circ \mathcal{L}^{n-1}_{m,p}(R) - I|$. That is, the convergence is of order $m$, and $\mathcal{L}_{m,p}$ exists for all $m$!

The transformations are based on the existence of rational functions $T_m$ such that $T_m(\cot \theta) = \cot m\theta$. As an example of a simple such transformation (that is, one of low order $m$) consider

$$I = \int_{-\infty}^\infty \frac{f x^2 + g x^2 + h}{ax^4 + bx^3 + cx^2 + dx + e} \,dx = \int_{-\infty}^\infty R(x)\,dx = \int_{-\infty}^0 R(x)\,dx + \int_0^\infty R(x')\,dx'$$

Let $y = T_2(x) = (x^2 - 1)/(2x)$. Solving for $x$ yields two branches $x_+(y) = y \pm \sqrt{y^2 + 1}$. Applying the changes of variable $x \rightarrow x(y), x' \rightarrow x(-y)$: $I = \int_{-\infty}^\infty (R_+(y) + R_-(y))\,dy$ where

$$R_+(y) = R(y + \sqrt{y^2 + 1}) + R(y - \sqrt{y^2 + 1}) \quad R_-(y) = \frac{y}{\sqrt{y^2 + 1}} \left( R(y + \sqrt{y^2 + 1}) - R(y - \sqrt{y^2 + 1}) \right).$$
However, it turns out that
\[ R_+(y) + R_-(y) = \frac{f'y^2 + g'y^2 + h'}{a'y^4 + b'y^3 + c'y^2 + d'y + e'} \]
is a rational function in \( y \) of a similar form to \( R \) and that the new coefficients are polynomials in the old ones. For example, \( a' = 16ae \) and \( h' = 2((f + h)(a + c + e) - g(b + d)) \). In fact, such is true of all \( \mathcal{L}_{m,p} \) and there is a deterministic way to directly generate these polynomials.

This suggests that in order to integrate a rational function on the real line, one might consider evaluating \( \pi \phi \circ \mathcal{L}_{m,p}^n(R) \) for sufficiently large \( n \). This indeed works, but there exists an issue with the exponential growth in the size of the coefficients under continued iterations and the increased complicatedness of \( \mathcal{L}_{m,p} \) for higher \( m \) that make exact computation unfeasible. A fixed-precision approximate decimal representation of the coefficients (perhaps after factoring out constant terms) is insufficient. The dynamical properties, in particular the stability, of \( \mathcal{L}_{m,p} \) are not fully understood for arbitrary \( m, p \). Therefore, no general understanding or bounding of the error that such a decimal approximation would entail is possible or hoped to be possible. However, our preliminary investigation of using small-order continued fraction convergents to approximate rational coefficients has proven successful. I hypothesize that such approximation holds promise for a tractable integration procedure that permits error analysis. With further investigation, I hope to produce some error bounds associated with such a procedure.

Another direction of investigation that might prove beneficial is seeking to use rational function approximations to general functions as a way to apply this integration procedure to general functions. In this case, our preliminary investigation has shown that the common Padé approximant is insufficient for such a purpose. However, I hypothesize that either an alternative way of approximation by rational function or another approach to the generalization of the technique will prove successful. With further investigation, I hope to work toward such a generalization of this exciting integration technique.

Some avenues of research that may appeal to beginning math students are improvements to the implementation, or perhaps a robust implementation in a language such as C. The search for helpful rational function approximations may also be appropriate for a beginning student. An open-ended project as that one would allow a beginning student to explore the world of math with increased independence: such is needed to develop understanding of mathematical research and confidence in one’s research abilities. Offering such projects to beginning students I hope to include others in my research activities thereby fostering learning and the inclusion of underrepresented groups in mathematical research as suggested in my personal statement.

The success of such an integration technique has important consequences. Fast numerical integration is critical in fields from physics to coding theory. And, following further work, this project may generate interdisciplinary collaboration as it draws on knowledge of pure math in its derivation, on knowledge of applied math in its error analysis, and on knowledge of computer science and engineering in its implementation and it finds uses in each discipline. As such it does well to enhance the infrastructure for research in multiple fields.
